Statistical Signal Processing
(2009-2010)
Handout #8: Hilbert-Huang Transform

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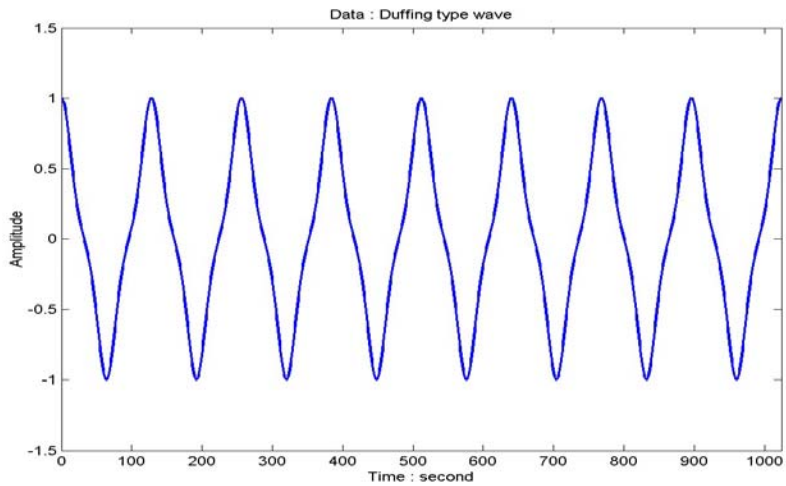
Outline

- A motivational example: Why Hilbert-Huang transform?
- Hilbert-Huang transform
  - The Empirical Mode Decomposition (EMD) method
  - Hilbert Spectral Analysis
- Comparison with Fourier transform and wavelet transform
- Further information

A motivational example

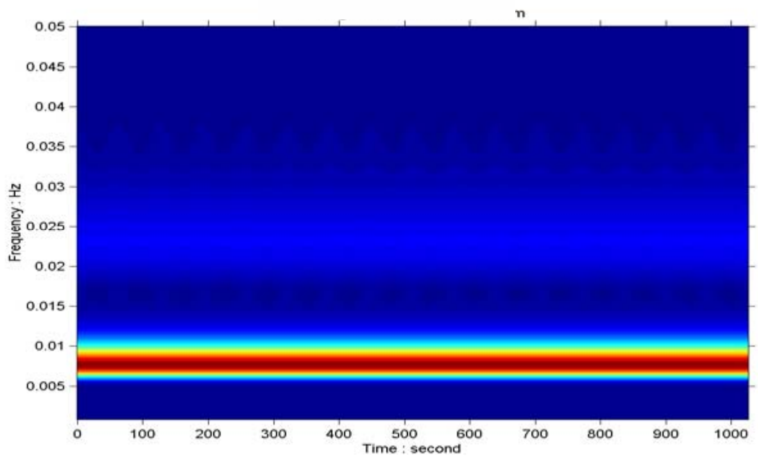
- x(t) = cos(wt+0.3 sin(2wt))



 See SSP_Handout8_HHT.m

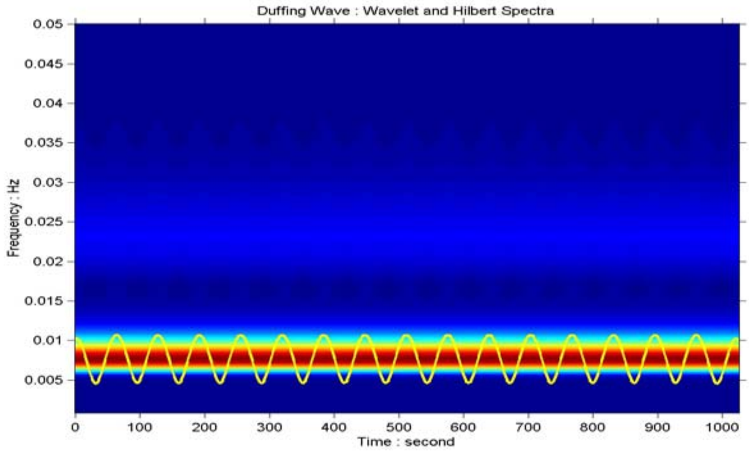
A motivational example

- Wavelet transform cannot exactly reveal how the frequency is varying with time.



A motivational example

- Hilbert-Huang transform deals with this problem successfully.

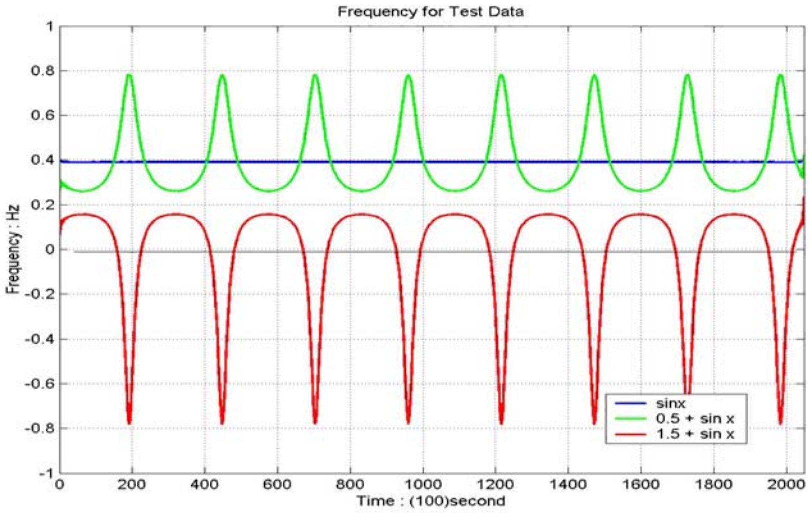


Hilbert-Huang Transform

- The HHT provides a new method of analyzing nonstationary and nonlinear time series data.
- The HHT consists of the Empirical Mode Decomposition (EMD) method (from Norden E. Huang) and Hilbert Spectral Analysis (HSA) (from David Hilbert (1862-1943)).
- In particular, the HHT uses the EMD method to decompose a signal into so-called intrinsic mode functions (IMFs), and uses the HSA method to obtain instantaneous frequency data.

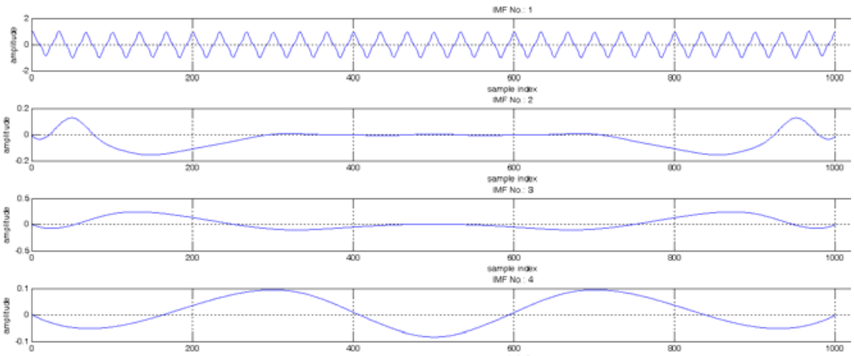
Why not use the HSA directly?

Hilbert-Huang Transform



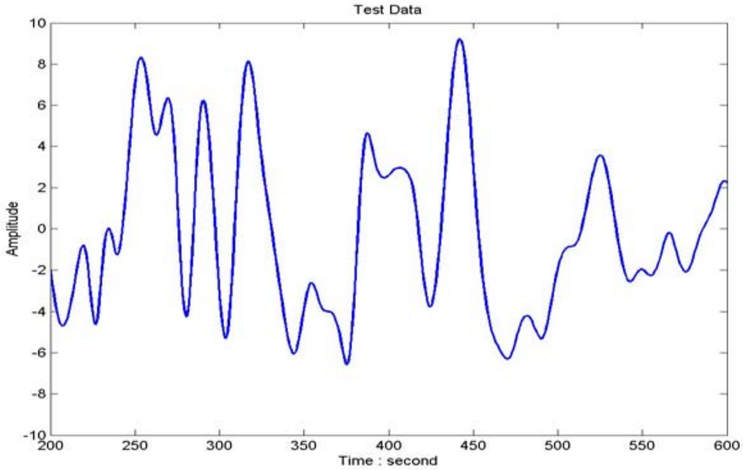
Hilbert-Huang Transform: The EMD method

- The EMD method is based on a simple assumption that any data consists of different simple intrinsic modes of oscillations. Each intrinsic mode, linear or nonlinear, represents a simple oscillation, which will have the same number of extrema and zero crossings.



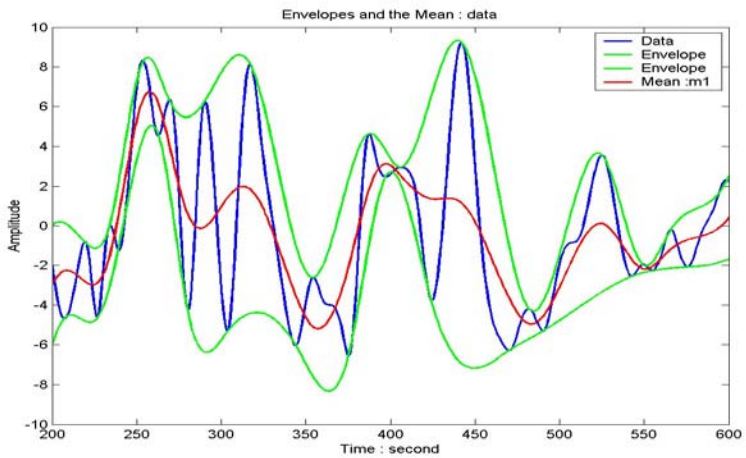
Hilbert-Huang Transform: The EMD method

- Original data



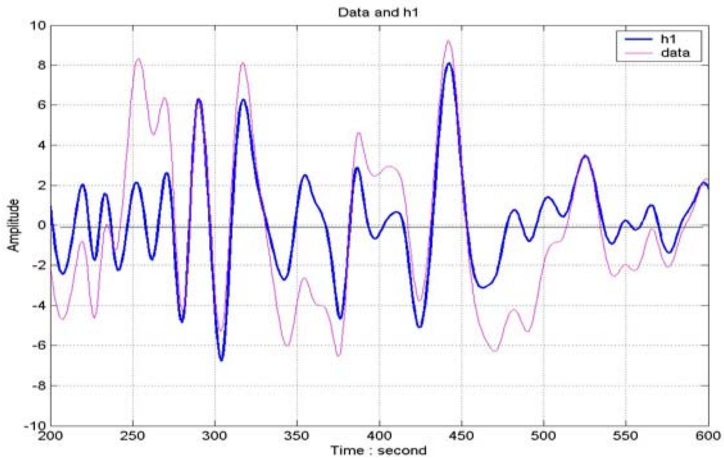
Hilbert-Huang Transform: The EMD method

- Identify all the local extrema, and connect all the local maxim (minima) by a cubic spline as shown the upper (lower) envelope.



Hilbert-Huang Transform: The EMD method

- The mean of the upper and lower envelope is designated as m_1, and the difference between the data $x(t)$ and m_1 is the first component $h_1 = x(t) - m_1$.



Hilbert-Huang Transform: The EMD method

- Repeat the same process till the resulted component h_k satisfies the requirements for the intrinsic mode function (IMF): the number of extrema and the number of zero-crossings much either equal or differ at most by one.

$$x(t) - m_1 = h_1,$$
$$h_1 - m_2 = h_2,$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$h_{k-1} - m_k = h_k.$$

$$\Rightarrow \quad h_k = c_1 \quad .$$

Hilbert-Huang Transform: The EMD method

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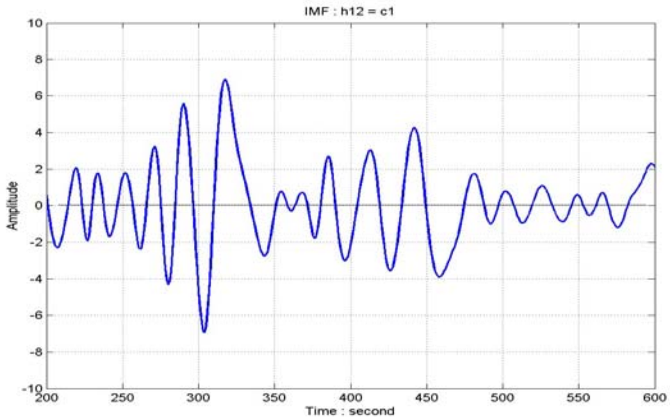


Hilbert-Huang Transform: The EMD method

- Repeat the above steps to obtain all the IMF components till no more IMFs can be extracted.

	IMF1	IMF2	IMF3		IMF n
	$X(t)$	$X(t)-c_1=r_1$	$r_1-c_2=r_2$		$r_{n-2}-c_{n-1}=r_{n-1}$
0	$X(t)-m_1=h_1$	$r_1-m_2=h_2$	$r_2-m_3=h_3$		$r_{n-1}-m_n=h_n$
1	$h_1-m_{11}=h_{11}$	$h_2-m_{21}=h_{21}$	$h_3-m_{31}=h_{31}$		$h_n-m_{n1}=h_{n1}$
2	$h_{11}-m_{12}=h_{12}$	$h_{21}-m_{22}=h_{22}$	$h_{31}-m_{32}=h_{32}$		$h_{n1}-m_{n2}=h_{n2}$
3	$h_{12}-m_{13}=h_{13}$	$h_{22}-m_{23}=h_{23}$	$h_{32}-m_{33}=h_{33}$		$h_{n2}-m_{n3}=h_{n3}$
..	……				
k	$h_{1(k-1)}-m_{1k}=h_{1k}$	$h_{2(k-1)}-m_{2k}=h_{2k}$	$h_{3(k-1)}-m_{3k}=h_{3k}$		$h_{n(k-1)}-m_{nk}=h_{nk}$
IMF	$h_{1k}=c_1$	$h_{2k}=c_2$	$h_{3k}=c_3$		$h_{nk}=c_n$

Hilbert-Huang Transform: The EMD method

- Repeat the above steps to obtain all the IMF components till no more IMFs can be extracted.

$$x(t) - c_1 = r_1 \ ,$$
$$r_1 - c_2 = r_2 \ ,$$
$$\cdots$$
$$r_{n-1} - c_n = r_n \ .$$

$$\Rightarrow x(t) - \sum_{j=1}^{n} c_j = r_n \ .$$

Hilbert-Huang Transform: The EMD method

- To be able to analyze data from the nonstationary and nonlinear processes and reveal their physical meaning, the method has to be Adaptive.
- Adaptive requires *a posteriori* (not *a priori*) basis. But the present established mathematical paradigm is based on *a priori* basis.
- Only *a posteriori* basis could fit the varieties of nonlinear and nonstationary data without resorting to the mathematically necessary (but physically nonsensical) harmonics.

Hilbert-Huang Transform: The HSA method

- The Hilbert transform of any real-valued signal is

$$H\left[x\left(t\right)\right]=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau$$

Here PV indicates the principle value of the singular integral.

Let $[\alpha,\beta]$ be a real interval and let f be a complex-valued function defined on $[\alpha,\beta]$. If f is unbounded near an interior point ζ of $[\alpha,\beta]$, the integral of f over $[\alpha,\beta]$ does not always exist. However, it may happen that the symmetric limit

$$\lim_{\epsilon\to0+}\left(\int_{\alpha}^{\zeta-\epsilon}f\left(x\right)dx+\int_{\zeta+\epsilon}^{\beta}f\left(x\right)dx\right)$$

exists. If it does, it is called the principle value of f from α to β, and is denoted as

$$PV\int_{\alpha}^{\beta}f\left(x\right)dx$$

Hilbert-Huang Transform: The HSA method

- With the Hilbert transform,

$$x\left(t\right)=A\left(t\right)\cos\Phi\left(t\right)$$

where

$$A\left(t\right)=\sqrt{\left(x\left(t\right)\right)^{2}+\left(H\left[x\left(t\right)\right]\right)^{2}}$$

$$\Phi\left(t\right)=\arctan\left(\frac{H[x(t)]}{x(t)}\right)$$

- The instantaneous frequency is the time derivative of the phase.

$$\omega=\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}$$

Hence, we have

$$x\left(t\right)=\mathrm{Re}\left\{A\left(t\right)e^{i\int\omega(t)\mathrm{d}t}\right\}$$

Hilbert-Huang Transform: The HSA method

- Given the period of a wave as T ; the frequency is defined as

$$\omega = \frac{1}{T}.$$

- The definition of instantaneous frequency is equivalent to defining velocity as

$$Velocity = \frac{distance}{time}\ ;\ mean\ \ velocity \qquad Newton \Rightarrow v = \frac{dx}{dt}$$

$$Frequency = \frac{1}{period}\ ;\ mean\ \ frequency$$

$$HHT\ defines\ the\ phase\ \ function \Rightarrow \omega = \frac{d\theta}{dt}$$

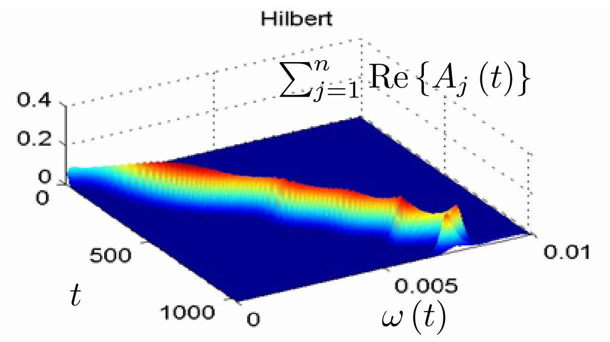
Hilbert-Huang Transform: The HSA method

- To be able to analyze data from the nonstationary and nonlinear processes and reveal their physical meaning, the method has to be local.
- Locality requires differential operation to define properties of a function.
- Take frequency for example. The present established mathematical paradigm is based on Integral transform. But integral transform suffers the limitation of the uncertainty principle.
- Instantaneous Frequency offers a total different view for nonlinear data: instantaneous frequency with no need for harmonics and unlimited by uncertainty.

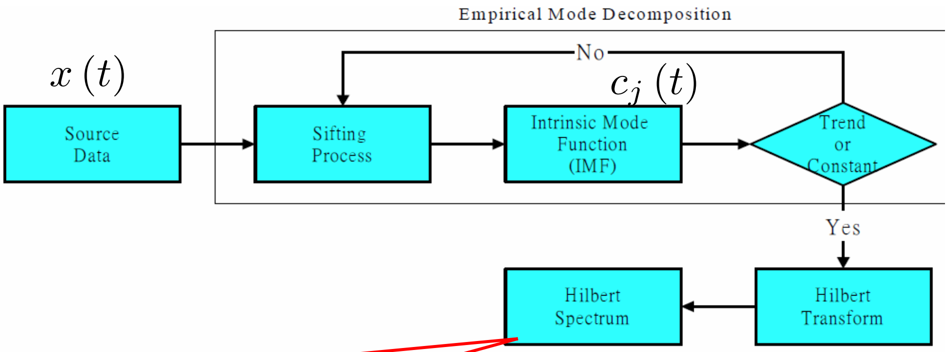
Hilbert-Huang Transform: The HSA method

- The HSA is the time-frequency representation of the original signal,

$$x(t) = \sum_{j=1}^{n} c_j = \sum_{j=1}^{n} \mathrm{Re}\left\{A_j(t)\, e^{i\int \omega(t)\mathrm{d}t}\right\}$$



Hilbert-Huang Transform: The computation flowchart



$$x(t) = \sum_{j=1}^{n} \mathrm{Re}\left\{A_j(t)\, e^{i\int \omega(t)\mathrm{d}t}\right\}$$

$$H[c_j(t)] = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{c_j(\tau)}{t-\tau} d\tau$$

Comparison

	Fourier	Wavelet	Hilbert
Basis	a priori	a priori	Adaptive
Frequency	Convolution: Global	Convolution: Regional	Differentiation: Local
Presentation	Energy-frequency	Energy-time-frequency	Energy-time-frequency
Nonlinear	no	no	yes
Non-stationary	no	yes	yes
Uncertainty	yes	yes	no
Harmonics	yes	yes	no

See SSP_Handout8_HHT.m for another example.

Further information

- N.E. Huang, A Plea for Adaptive Data Analysis: An Introduction to HHT *(presented by Dr. Huang at PKU in the summer of 2008)*
- C.M. Hsieh, Hilbert Huang Transform, 2007 (in Chinese).
- N.E. Huang, Introduction to the Hilbert-Huang transform and its related mathematical problems, Chapter 1 in Hilbert-Huang Transform and Its Applications, N.E. Huang and S. P. Shen (Eds), World Scientific Publishing Company 2005. *(A short but well-written introductory-level article on HHT).*
- A comprehensive website including Matlab program, papers, course materials at http://rcada.ncu.edu.tw/research1.htm