Time-Frequency Analysis and Wave Transform Term Paper

Tutorial

Hilbert Huang Transform for Climate Analysis

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Abstract

Climate is a complicated system containing four topics which are temperature, rainfall, atmospheric pressure and wind. Analyzing climate well can help people predict weather situation and avoid weather hazards. In the recent years, climate changing causing many disasters such as floods, droughts and heat waves, make a tremendous threat to human beings. Therefore, we need a strongly mathematic model to deal with such large data and abstract useful information. Several linear mathematical models have been applied to climate data, but the results are not so decisive and crucial because climate data are non-linear and non-stationary time series data. Fortunately, an adaptive mathematic model, the Hilbert-Huang transform (HHT), developed by Huang recently seems to be able to solve the problem. In this paper, the first part is an introduction to the Hilbert-Huang transform, and the second part is the result and discussion of experiment.

1. Hilbert-Huang Transform

In this paragraph, we will make an introduction to the Hilbert-Huang transform followed by the brief description of the Hilbert transform, Hilbert spectrum, intrinsic model function and empirical model function. The Hilbert-Huang transform is composed of these parts, and then some mathematic problems associated with the Hilbert-Huang transform are discussed.

1.1. Introduction

Most traditional analysis methods are based on linear and stationary assumption [2]. For example, the wavelet analysis, the Wigner distribution and the STFT were designed for linear but non-stationary data. Additionally, various non-linear time series analysis was designed for non-linear but stationary and deterministic systems. However, in either natural or anthropogenic environment, most systems are both non-linear and non-stationary. For non-linear and non-stationary systems, adaption is definitely necessary. Adaption means that the mathematic model has to be data-dependent. The biggest difference from the Hilbert-Huang transform and traditional analysis methods is that the Hilbert-Huang transform is an empirically based data-analysis method. The Hilbert-Huang transform is an adaptive way to produce physically meaningful representation of data from non-linear and non-stationary system, especially for time-frequency-energy representation. Powerful as the Hilbert-Huang transform is, the advantage of being adaptive has a price. There is no complete theoretical foundation. In order to make the Hilbert-Huang transform more robust, there exists some mathematical problem to be solved. Some of the problems have been solved, but the other parts still need more effort to be carried on.

1.2. The Hilbert-Huang Transform

This new approach, the Hilbert-Huang transform, is developed by the need to analyze the non-linear and non-stationary data. Periodicity is a typical characteristic of non-stationary process, and we can derive it from its intra-wave frequency form, which indicates the instantaneous frequency changes within one oscillation cycle. As an example, a simple non-linear system is showed, given by the non-dissipative Duffing equation as

$$\frac{d^2x}{dt^2} + x + \varepsilon x^3 = \gamma \cos(\omega t) \tag{1.1}$$

Where ε is a parameter not necessary small, and γ is the amplitude of a periodic forcing function with frequency ω. If the parameter ε is zero, the system turns into a linear system. However, if the parameter ε is not zero, the system would be a non-linear system. More importantly, if the parameter ε is not small enough, it would cause bifurcations and chaos, and then this equation is not useful. By rewriting the equation into a slightly different form as

$$\frac{d^2x}{dt^2} + x(1 + \varepsilon x^2) = \gamma \cos(wt) \tag{1.2}$$

In this form, the quantity within the parenthesis can be considered as a variable spring constant or a variable pendulum. In the pendulum system, frequency changes from time to time, and location to location. In 1998, Huang et al. found that intra-wave frequency variation is crucial to non-linear system. In the traditional Fourier transform, the intra-wave frequency variation cannot be depicted, except by resorting to harmonic. Thus, any non-linear distorted waveform can be regarded as "harmonic distortions." Harmonic distortions are a mathematic artificial consequence of imposing a linear structure on a non-linear system. They may have mathematical meanings but physical meanings (Huang et al. 1999). Therefore, the physically meaningful way to describe a non-linear system is instantaneous frequency, which will reveal the intra-wave frequency modulations.

The Hilbert transform is the easiest way to compute instantaneous frequency, through which the complex conjugate $y(t)$ of any real value function $x(t)$ of L^p class can be determined by

$$y(t) = H[x(t)] = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \tag{1.3}$$

In which the PV indicates the principle value of singular integral, and the analytic signal is defined as

$$z(t) = x(t) + iy(y) = a(t)e^{i\theta(t)} . \tag{1.4}$$

Where

$$a(t) = \sqrt{x^2 + y^2} \text{ , and } \theta(t) = arc\tan(\frac{y}{x}) , \tag{1.5}$$

$a(t)$ is the instantaneous amplitude, and $\theta(t)$ is the phase function, and the instantaneous frequency is simply

$$\omega = \frac{d\theta}{dt} . \tag{1.6}$$

In fact, using the Hilbert transform directly would cause a problem, which a sensible instantaneous frequency cannot be found. As a result, the Hilbert transform was applied to narrow band-passed signal, which is narrow-banded with the same number of extrema and zero-crossings. However, filtering a signal into a narrow-banded signal is a linear operation, so filtered data will be stripped of their harmonics, and the result will be a distortion of a waveform. The filtering process is called the empirical mode decomposition method (EMD).

1.2.1. ***The Empirical Mode Decomposition Method (The Sifting Process)***

Compared to traditional analysis method, the Hilbert-Huang transform is intuitive, direct, and adaptive, with *a posteriori*-defined basis, from the decomposition method, based on and derived from the data. The decomposition has an assumption that any data consists of different simple intrinsic models of oscillations. Each intrinsic mode, no matter linear or not, represents an oscillation, which will have the same number of extrema and zero-crossings, and then the oscillation will be symmetric with respect to the local mean. Usually, the data may have many different oscillations which can be represented by the intrinsic mode functions (IMF) with following definition:

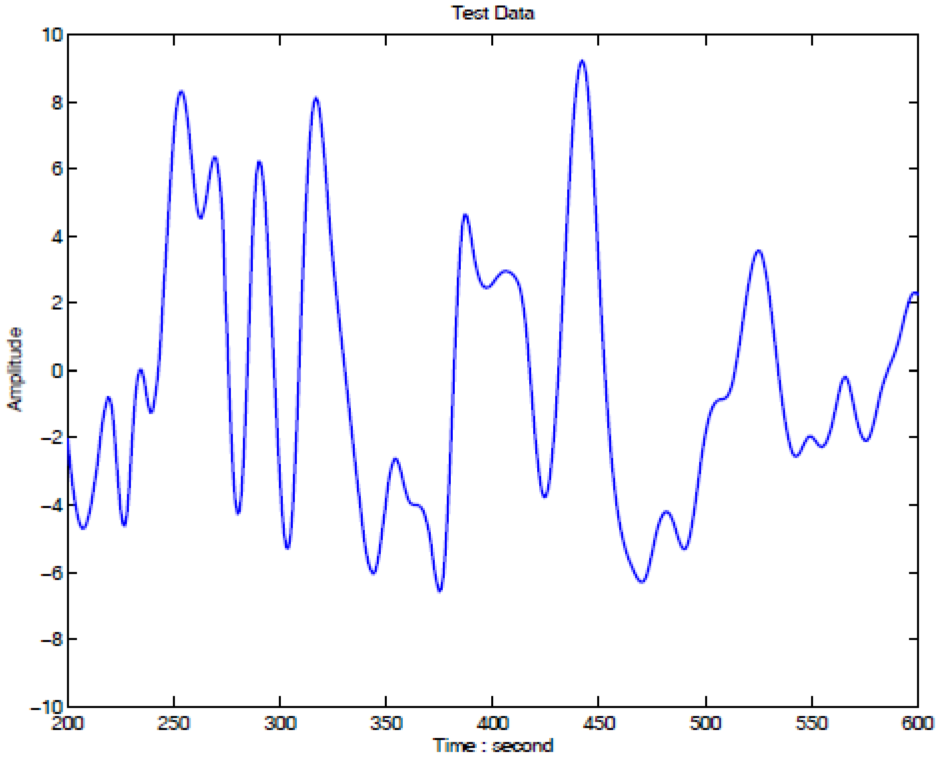

Figure 1.1: The test data.

(a) in the whole dataset, the number of extrema and the number of zero-crossings must either equal or differ at most by one, and

(b) at any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero.

An IMF is much more general than an oscillation mode because it has a variable amplitude and frequency as a function of time. According to the definition for the IMF, we can decompose any function as follows [4] and take Fig. 1.1 as an example.

(1) First, find all the local maxima extrema of $x(t)$

(2) Interpolate (cubic spline fitting) between all the maxima extrema ending up with some upper envelope $e_{\max}(t)$.

(3) Find all the local minima extrema.

(4)Interpolate (cubic spline fitting) between all the minima extrema ending up with some lower envelope $e_{\min}(t)$.

(5) Compute the mean envelope between upper envelope and lower envelope shown in Fig. 1.2

$$m_1 = \frac{e_{\min}(t) + e_{\max}(t)}{2} \tag{1.7}$$

(6) Compute the residue h_1 shown in Fig. 1.3

$$h_1 = x(t) - m_1 \tag{1.8}$$

(7) Here, a critical decision must be made: the stoppage criterion. If this squared difference SD_k is smaller than a predetermined threshold, the sifting process will be stopped.

$$SD_k = \sum_{t=0}^{T} \left[\frac{\left|(h_{1(k-1)}(t) - h_{1k}(t))\right|^2}{h_{1(k-1)}^2(t)} \right] \tag{1.9}$$

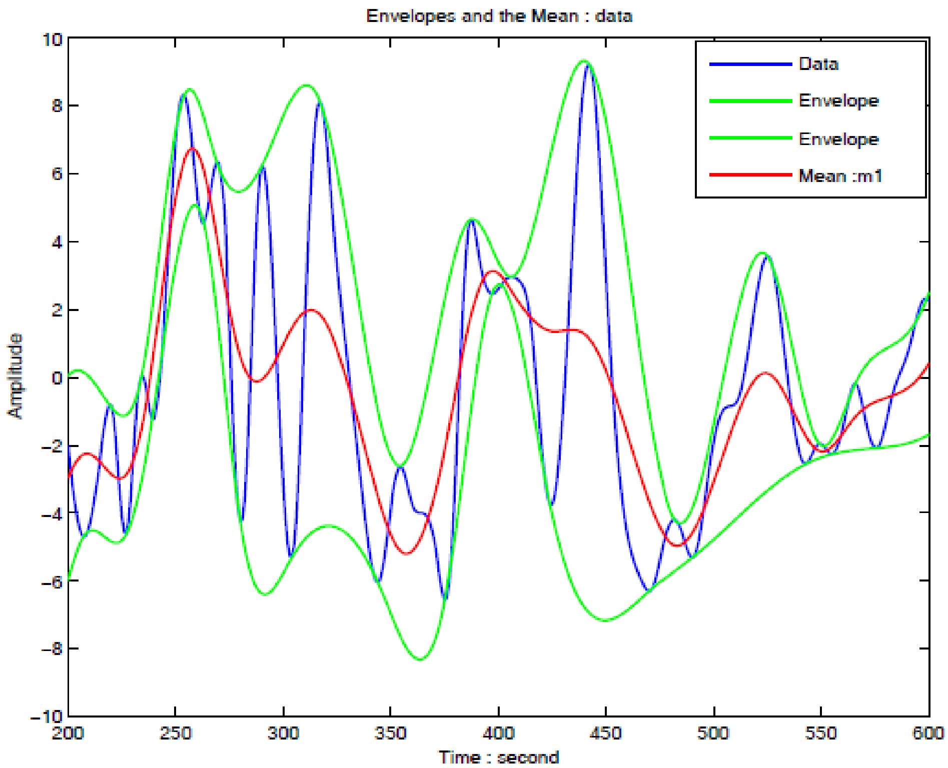

Figure 1.2: The data upper and lower envelopes defined by local maxima and minima, and the mean value of two envelopes.

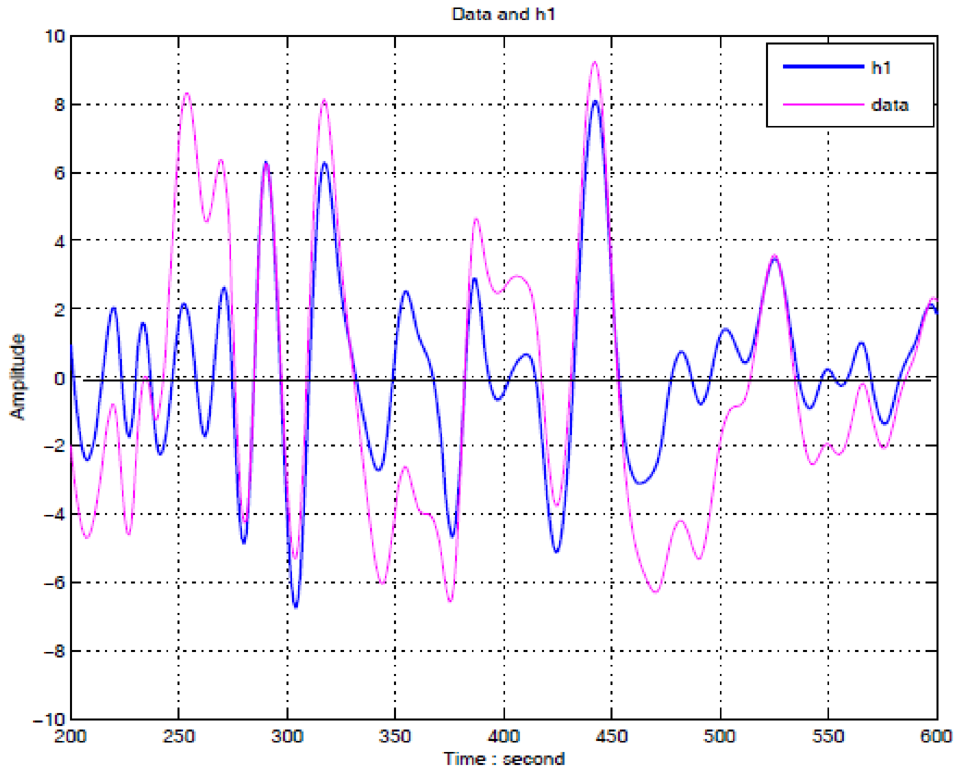

Figure 1.3: The data and h_1.

Ideally, h_1 should satisfy the definitions of an IMF, so it should be symmetric and have all maxima positive and all minima negative. However, the hump on slope may become a local maximum after the first round of sifting, and then the residue may not satisfy the definitions of an IMF. The sifting process has two purposes: to eliminate riding waves, and to make the wave profiles more symmetric. The first purpose is designed for the Hilbert transform to give a meaningful instantaneous frequency, the second purpose is designed in case the neighboring wave amplitude have too large disparity. For these two purposes, the sifting process should be repeated until to extract the residue satisfying the definition of an IMF. In the next step, h_1 is treated as a new data; then,

$$h_{11} = h_1 - m_{11} \tag{1.10}$$

After repeating sifting process, show in Fig. 1.4 (a) and (b), up to k times, h_{1k} becomes an IMF; that is,

$$h_{1k} = h_{1(k-1)} - m_{1k} \tag{1.11}$$

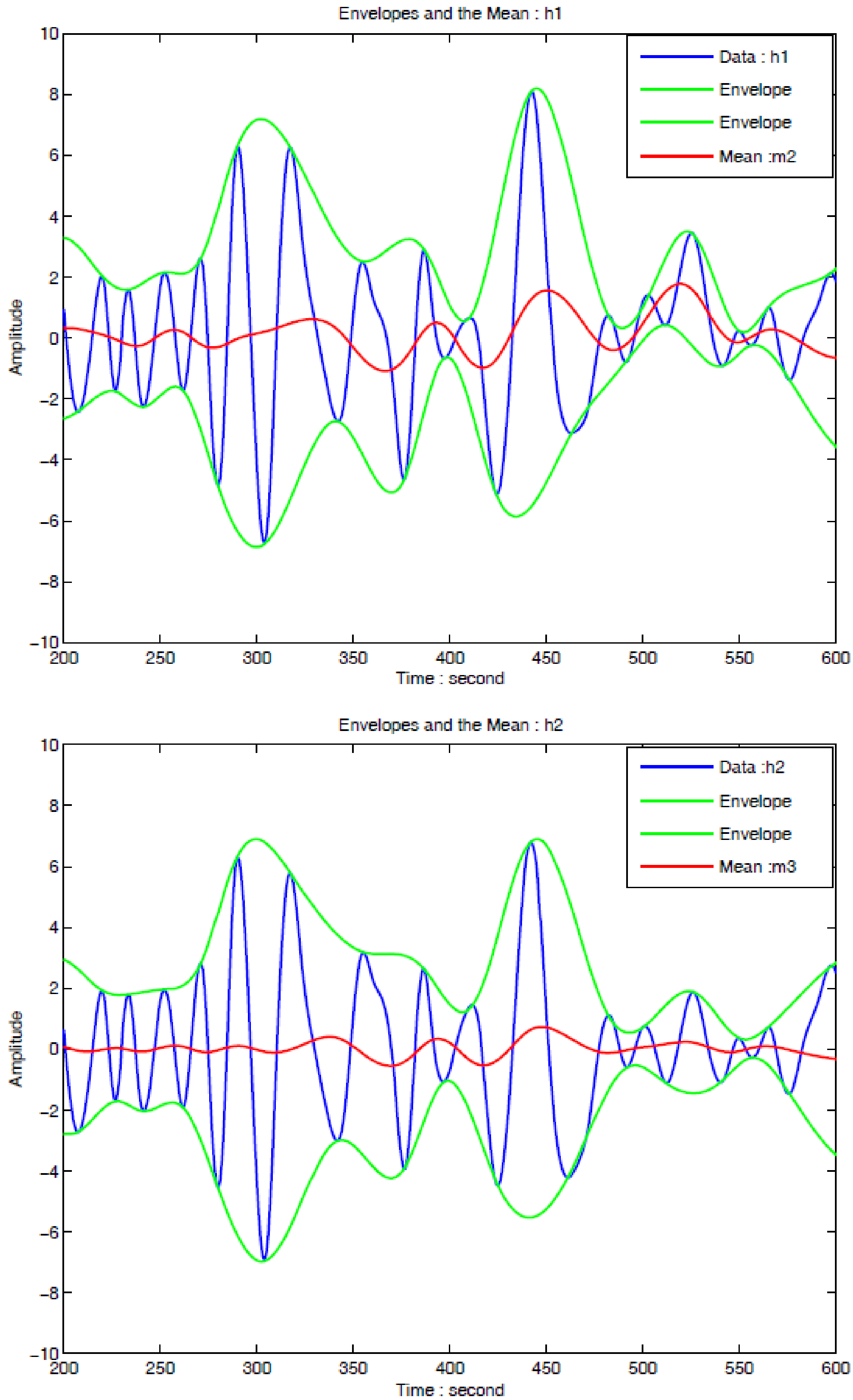

Figure 1.4: (a,top) Repeated sifting steps with h_1 and m_2. (b,bottom) Repeated sifting steps with h_2 and m_2.

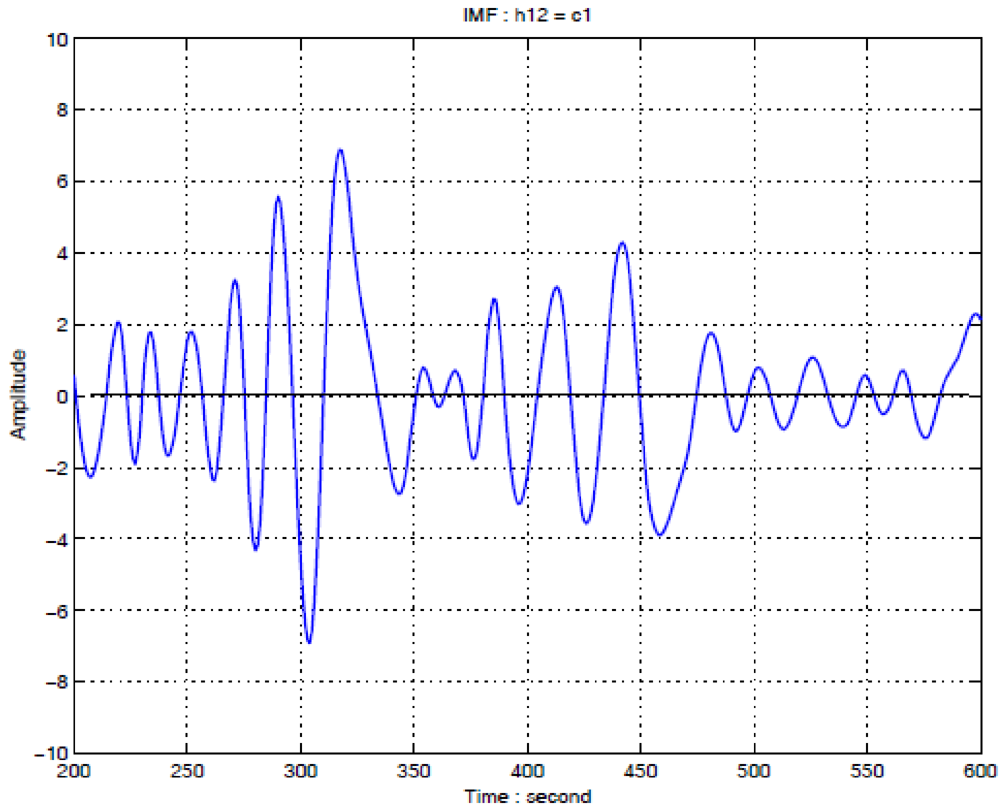

Figure 1.5: The first IMF component c_1 after 12 steps.

Then, it is designated as

$$c_1 = h_{1k} \tag{1.12}$$

the first IMF component from the data shown in Fig. 1.5.

However, there are two critical problems needed to be solved: first, the problem how small is small enough needs an answer. Second, this criterion does not depend on the definition of an IMF. These two problems mean that nothing guarantees that the function will have the same number of zero-crossings and extrema. If readers are interested in how to decide times of sifting steps, there are many researches developed.

Now we assume that a stoppage criterion was selected, and that the first IMF c_1 was found. c_1 should contain the finest scale or the shortest period component of the signal.

The rest of the data is that,

$$r_1 = x(t) - c_1 \tag{1.13}$$

Since the residue r_1 still contains longer period variations in the data, as show in Fig. 1.6, it is treated as the new data and we repeat the same sifting process to it. The result is

$$\begin{aligned} r_2 &= r_1 - c_2 \\ &\vdots \\ r_n &= r_{n-1} - c \end{aligned} \qquad (1.14)$$

Fig 1.7 shows the flowchart of IMF computation. No matter how small the component c_n or the residue r_n is, or no more IMFs can be extracted, the final residue still can be different from zero. If the data have a trend, the final residue should be that trend. By summing up (1.12) and (1.13), we obtain that

$$x(t) = \sum_{j=1}^{n} c_j + r_n \quad . \qquad (1.15)$$

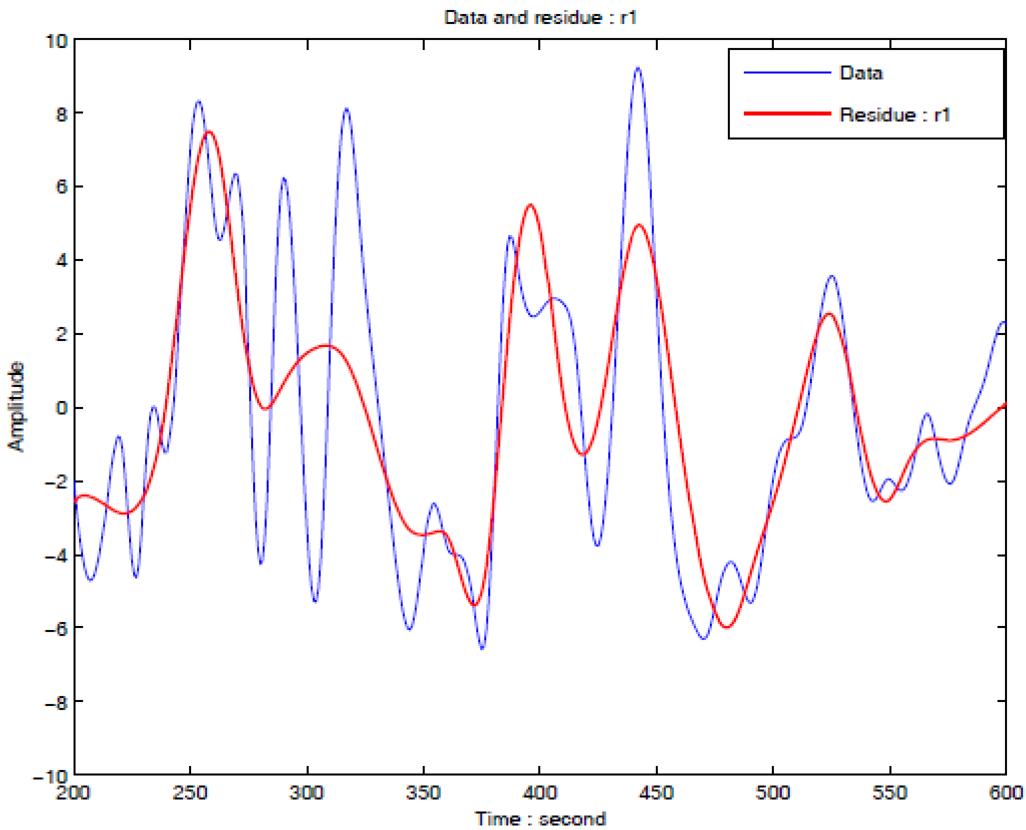

Figure 1.6: The original data and the residue r_1

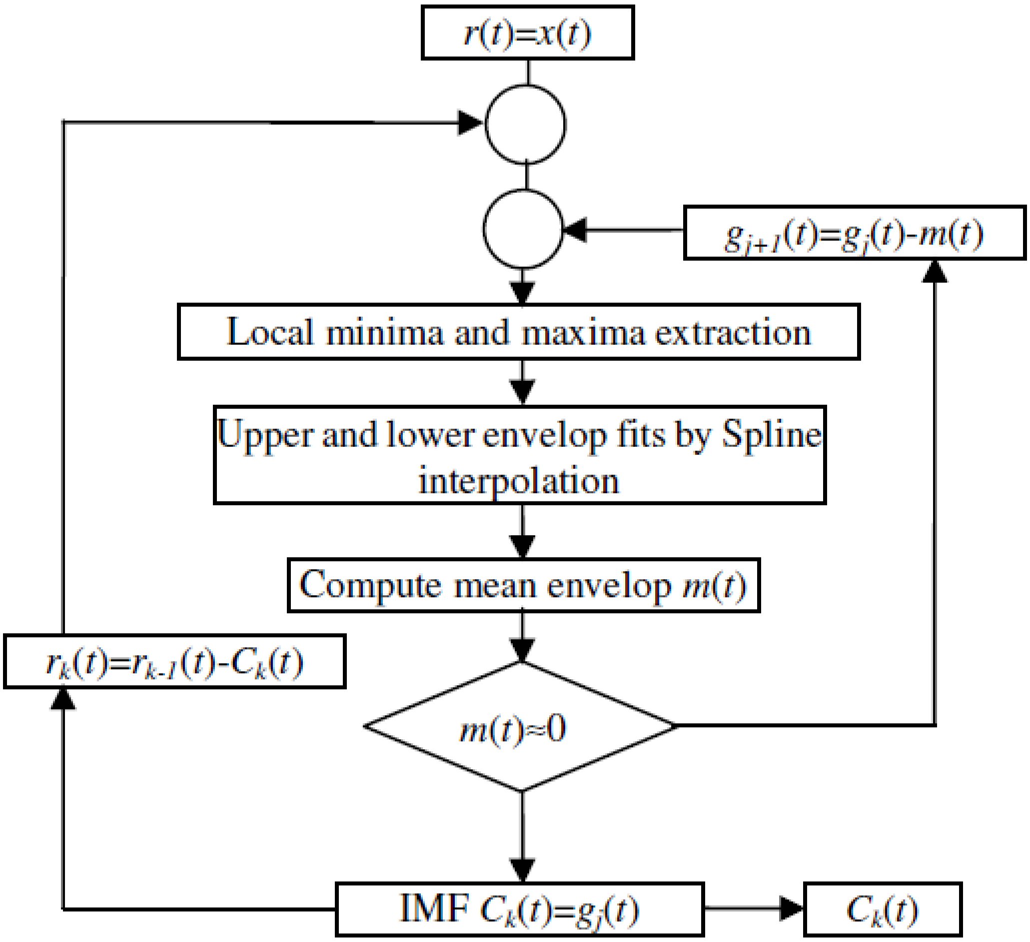

Figure 1.7: The flowchart of EMD

Thus, an n-empirical mode decomposition is achieved, and the residue r_n is obtained, which can be either the mean trend or a constant. The components of the EMD are usually physically meaningful, for the characteristic scales are defined by the physical data. In the second part, the aspect will be introduced through the temperature changing in the climate.

1.2.2. *The Hilbert Spectral Analysis*

After the empirical mode decomposition, we can apply the Hilbert transform to each IMF component, and compute the instantaneous frequency according to (1.3)-(1.6). Consequently, the original data can be expressed as the real part in the following form:

$$x(t) = R\{\sum_{j=1}^{n} a_j(t)\exp[i\int \omega_j(t)dt]\} \quad . \tag{1.16}$$

There are two reason that the residue r_n should be left out. First, the energy involved in the residual trend representing a mean offset could be overpowering. Second, we are more interested in obtaining the information contained in the other low-energy but clearly oscillatory components rather than the uncertainty of the longer trend.

Equation (1.16) can be modified into a Fourier representation as

$$x(t) = R[\sum_{j=1}^{n} a_j e^{i\omega_j(t)t}] \tag{1.17}$$

with both a_j and ω_j as constants. In equation (1.17), the variable amplitude and the instantaneous frequency have not only improve the efficiency of expansion, but also enabled the expansion to accommodate non-linear and non-stationary data. Therefore, the restriction of the constant amplitude and fixed frequency of the traditional Fourier expansion have been conquered, with a variable amplitude and frequency form, which is called "Hilbert amplitude spectrum" or "Hilbert spectrum". If amplitude is squared, we obtain the Hilbert energy spectrum.

With the Hilbert Spectrum defined, we can also define the marginal spectrum h(ω) as

$$h(\omega) = \int_0^T H(\omega,t)dt \quad . \tag{1.18}$$

The marginal spectrum offers a measure of the total energy distribution from each frequency.

The combination of the empirical mode decomposition and the Hilbert spectral analysis is known as the Hilbert-Huang transform. Different from the traditional analyses, the Hilbert—Huang transform is developed for non-linear and non-stationary data. There is a comparative summary of the Fourier transform, the wavelet transform, and the Hilbert-Huang transform given in the following table:

Table 1.1: The comparison between the HHT, the wavelet transform, and the FT

	Fourier	Wavelet	Hilbert
Basis	a priori	a priori	Adaptive
Frequency	convolution: global uncertainty	convolution: regional uncertainty	differentiation: local certainty
Presentation	energy-frequency	energy-time-frequency	energy-time-frequency
Non-linear	no	no	yes
Non-stationary	no	yes	yes
Feature extraction	No	discrete: no continuous: yes	yes
Theoretical base	theory complete	theory complete	empirical

According to Table 1.1, we found that the Hilbert-Huang transform is a powerful tool to analyze data from non-linear and non-stationary system. The Hilbert-Huang transform is an adaptive process and the instantaneous frequency is derived by differentiation rather than convolution. Therefore, the Hilbert-Huang transform is not limited by uncertainty principle. Besides, the Hilbert-Huang transform is used for feature extraction because it can represent result in time-frequency-energy distribution.

1.3. Mathematical problems related to related the HHT

The Hilbert-Huang transform is developed in 1999. However, there are still many mathematical and theoretical problems to be solved to this day. Here are some main problems as following:

a. Adaptive data analysis methodology in general
b. Non-linear system identification methods
c. Spline problems
d. Optimization problems

1.3.1. Adaptive Data-Analysis Methodology

As we discussed at the beginning, the traditional analysis methods are designed for linear and stationary data. In this way, there are many restrictions for applying these methods to man-made or natural systems because most systems are non-linear and non-stationary. Therefore, we have to design a analysis method based on data. Under this paradigm, there is no solid foundation which has been developed until today. In the recent years, more and more scientists pay attention to adaptive analysis methods and developed some methods. However, most of them depend on feedback, so they are confined to stationary processes. As the result, how to design an adaptive model is still challenging.

1.3.2. Non-linear System Identification

In traditional way, we compute the relation between input and output to identify whether the system is linear or not. For some ideal cases, it may work out. However, in the most man-made and natural systems, we cannot control input and output. The only data we have is the output from an unidentified system. In the above lecture, Huang indicated that an IMF may be an index for nonlinearity. However, it is still not enough to distinguish a quasi-linear from a truly nonlinear system. An effective method for nonlinear-system identification is urgently needed.

1.3.3. Spline Problems

In the sifting process, we use spline fitting to connect local maxima and local minima. However, among all the spline methods, which one is the best for the interpolation? The answer is critical. In the experiment, cubic spline applied to the sifting process is more appropriate. If lower-order spline is used, the envelop is not smooth and cling enough. If higher-order spline is used, it is against the spirit of adaptation. For adaptation, we try not to add too many parameters to analysis models. Higher-order spline needs more parameters and costs more computation time, so it is not suitable.

1.3.4. *The Optimization Problem*

As we discusses, if we choose different splines, we will get different sets of IMFs. If we use different stoppage criterion, we will get different sets of IMFs. What is the relation between these sets? Which set contains more physical meaning? Consequently, how to optimize the sifting process to get the best IMF set is an important question.

2. Analysis of Climate by the HHT

Climate is a complicated natural system, which contains temperature, wind, atmospheric pressure, and rainfall. Each of these four topics plays an important role in the global environment. In this paragraph, we take temperature change as the experiment using the Hilbert-Huang transform.

2.1. *Introduction*

Although people have researched climate for many years, climate as a complex system still challenges our knowledge, leaving us with the problems that deal with sparse data, insufficient methods, limited models, and unexplained physical processes [3]. Nowadays, global warming and greenhouse effect are most popular issues causing huge threat to lives and properties. The global mean temperature has increased by $0.6^{o}C$ over the last century and resulted in many changes to the human and animal community. In the recent years, researchers found that global warming is the reason of some natural disasters, such as floods, drought, and heat waves which are more frequent than ever. Therefore, people and government have spent much resource on climate analysis. Several linear statistical models have been applied to climate data, but the answer is not satisfied because of the nonlinearity and non stationary of

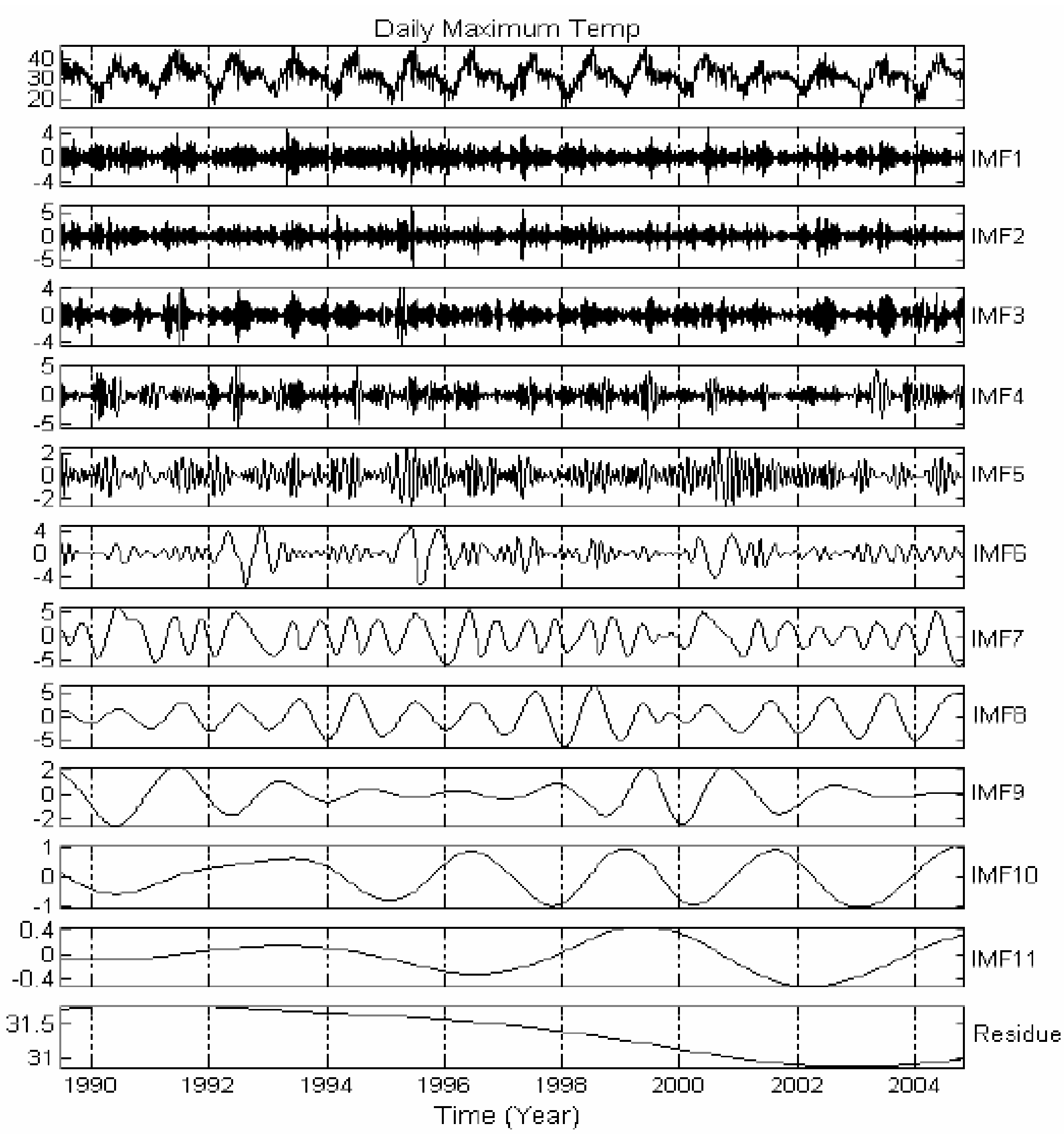

Figure 2.1: IMF components and final residue of daily maximum temperature

climate data. A new analysis method, the Hilbert-Huang transform which is designed for non-linear and non-stationary data, is applied to climate data. Although it lacks of full theory, the Hilbert-Huang transform still offers a better understanding of the variability of the region climate measured by the changes in the surface air temperature.

2.2. Experiments and Discussion

The empirical mode decomposition method is applied to the 15 years daily maximum and minimum temperature data. The EMD results are shown in the following Fig. 2.1 and Fig. 2.2. These two figures show that strong inter-annual

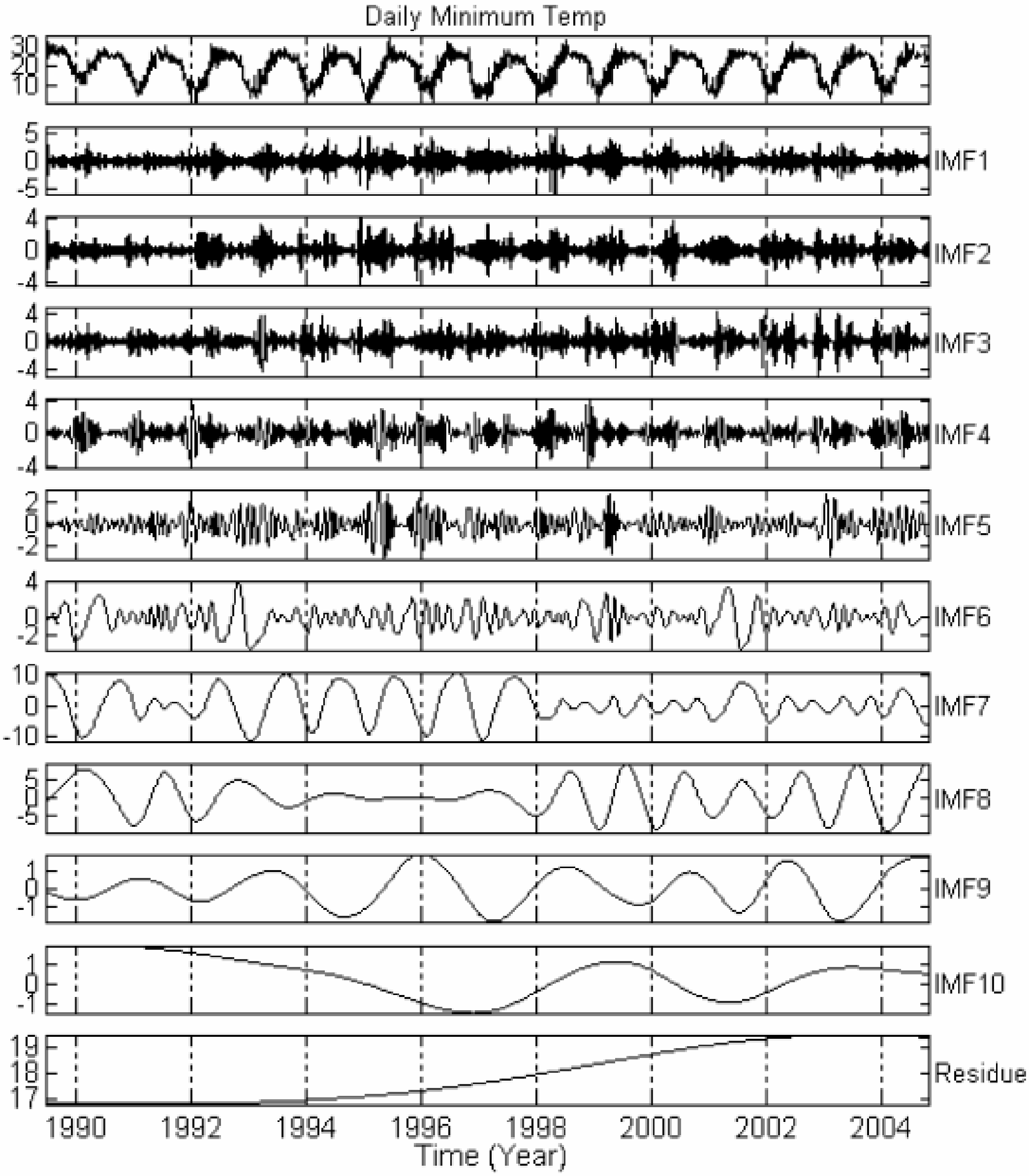

Figure 2.2: IMF components and final residue of daily minimum temperature

fluctuations exists both the maximum and minimum temperature. The residues indicate that the trend of daily maximum temperature is going down, and the trend of daily minimum temperature is going up. The residual series are slightly dependent which may be due to small underlying trends caused by climate change or El Nino, which happens once every four years. After EMD, each IMF is applied to the Hilbert spectral analysis to calculate the instantaneous frequency. The Hilbert spectrum is a time-frequency-energy distribution, and it is easy to clearly observe the frequent and non-frequent temperature change at any time over the entire data length. The most popular form to present the Hilbert spectrum is the color map presentation corresponding to the energy in dB shown in Fig. 2.3.

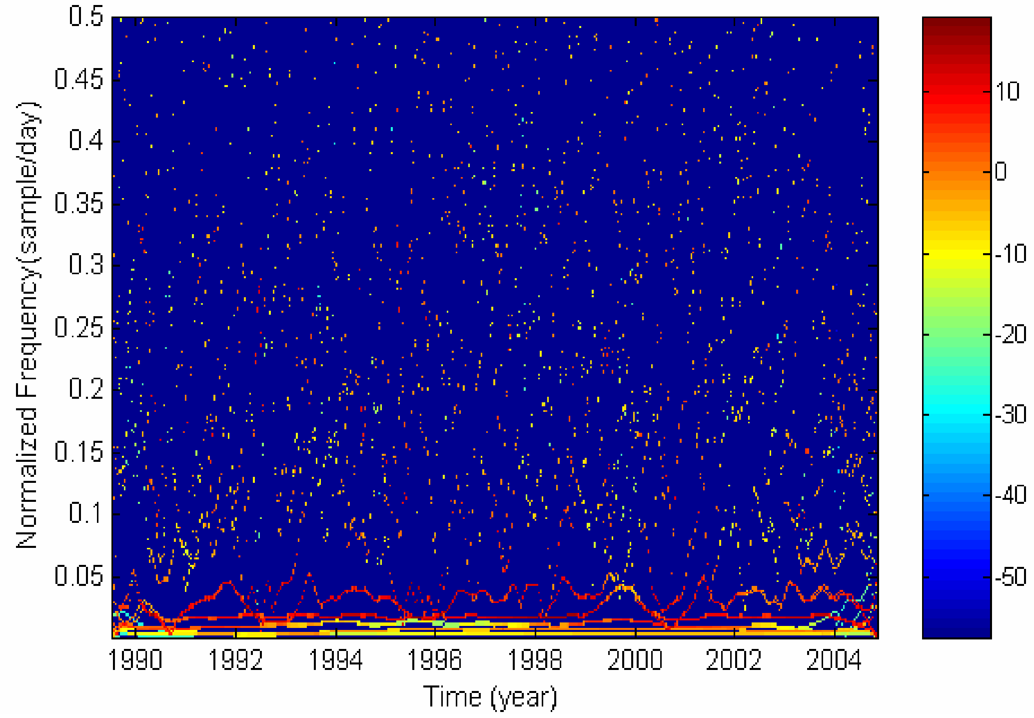

Figure 2.3a: Hilbert spectrum of maximum temperature

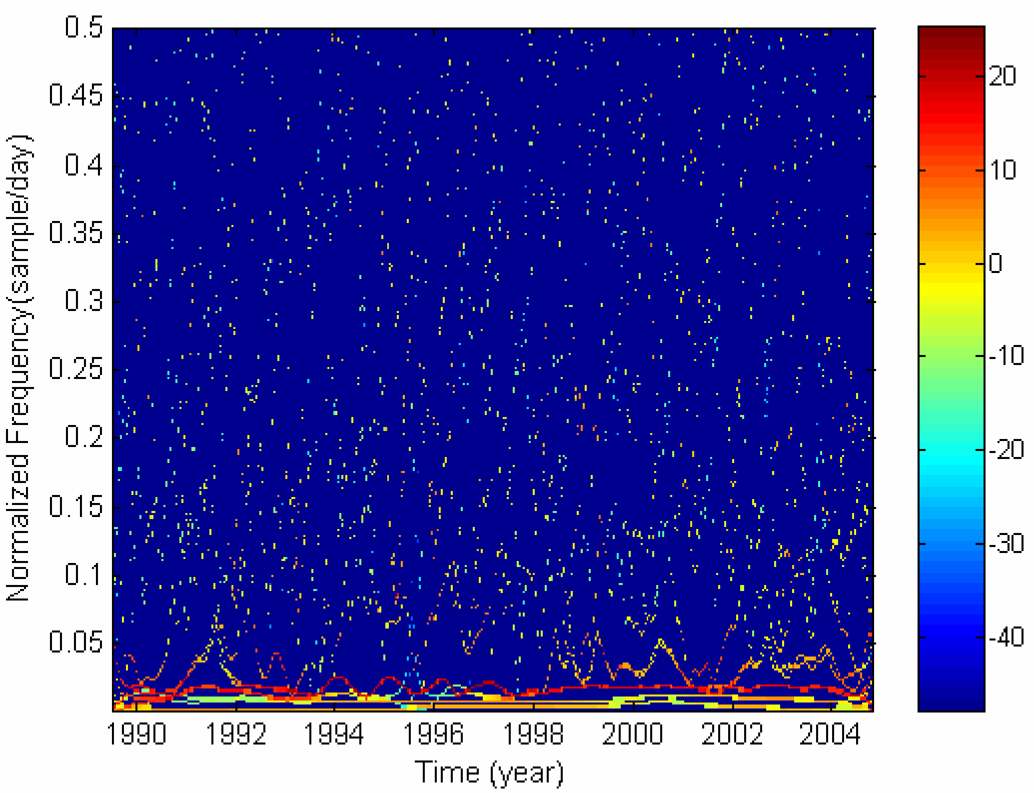

Figure 2.3b: Hilbert spectrum of minimum temperature

2.2.1. IMF Component and its Probability Distribution

In the marginal spectrum, the energy at the frequency ω means there is a higher likelihood that an oscillation with such a frequency exists. The probability distribution of individual IMFs is shown in Fig. 2.4. According to the Central Limit Theorem, the probability density function is approximately normal distribution when the number of sample is large. The deviation grows as the mode number increases because there are fewer oscillations in the higher frequency modes. The IMFs isolate physical processes of various time-scales and also give the temporal variation without resorting to the linear assumption. Therefore, IMFs can be effectively used to construct the time-frequency distribution in the form of Hilbert spectrum.

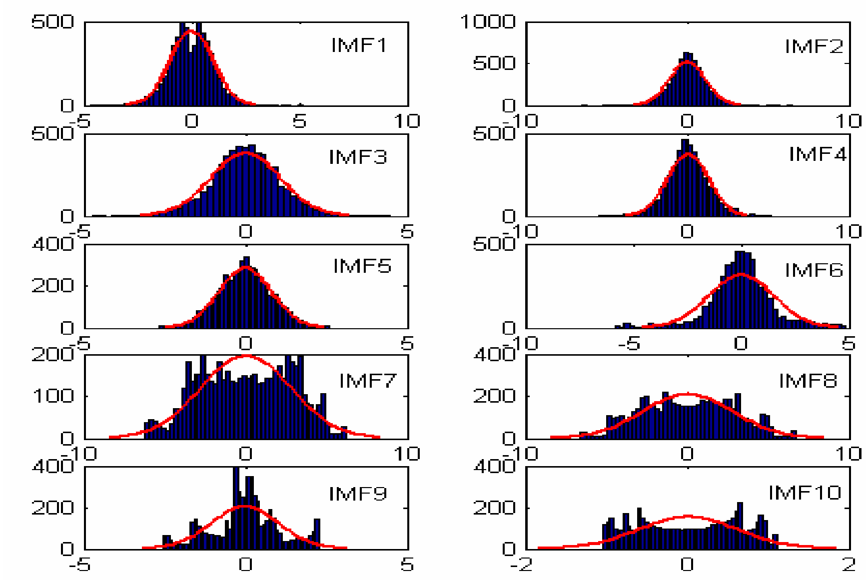

Figure 2.4a: Probability distribution of the IMFs of maximum temperature

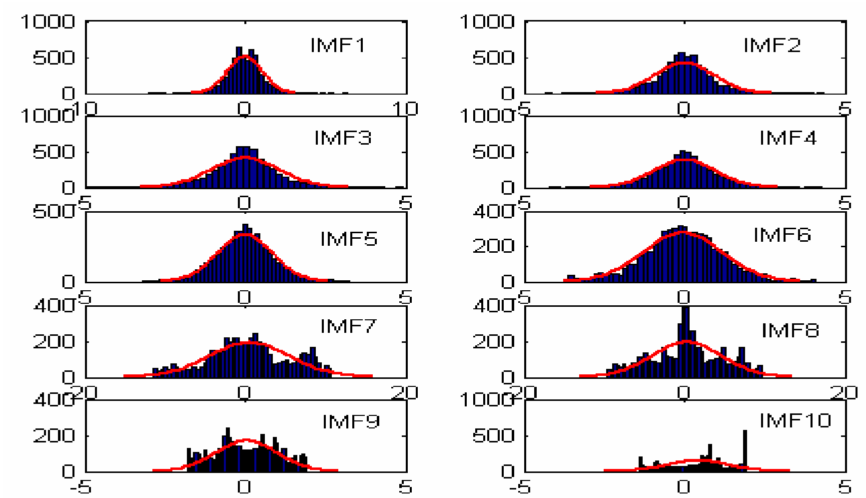

Figure 2.4b: Probability distribution of the IMFs of minimum temperature

2.2.2. Reconstruction, Orthogonality and Correlation of IMFs

As shown in (1.15), data can be linearly decomposed into a set of IMFs if the EMD method is ideal. Therefore, data can be reconstructed by adding of all IMF components with negligible error given in Fig. 2.5.

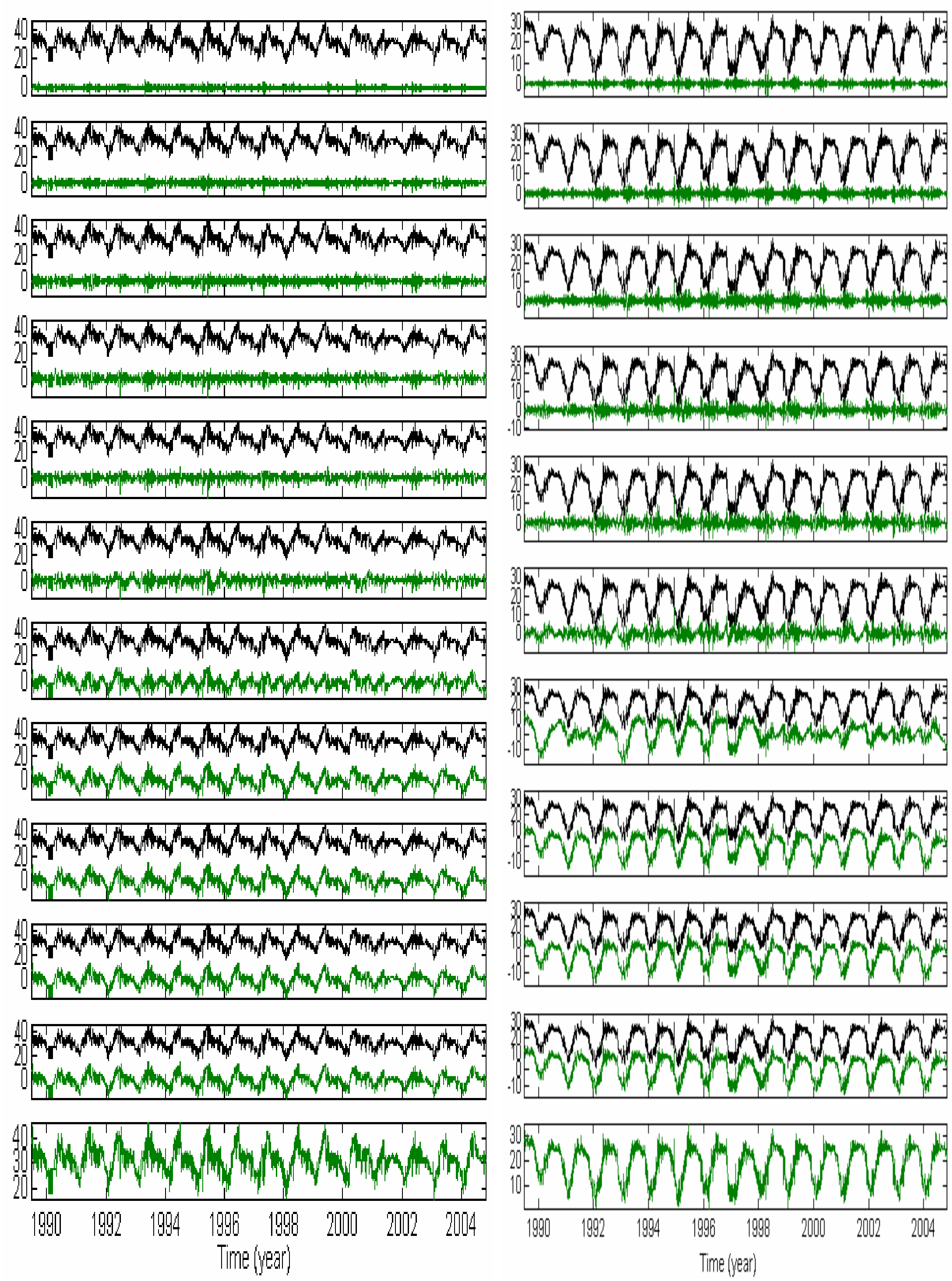

Fig. 2.5: Reconstruction from IMFs of maximum and minimum temperature; the black one is the original data and green one is the stepwise reconstruction.

The IMFs from an efficient EMD method should be approximately orthogonal to each other. Fig. 2.6 shows orthogonality values between all pairs of IMFs of maximum and minimum temperature. The x-axis and the y-axis represent the indices of IMFs, and the z-axis represents the index of orthogonality. In other words, orthogonality can be used to detect whether the EMD is efficient enough or not.

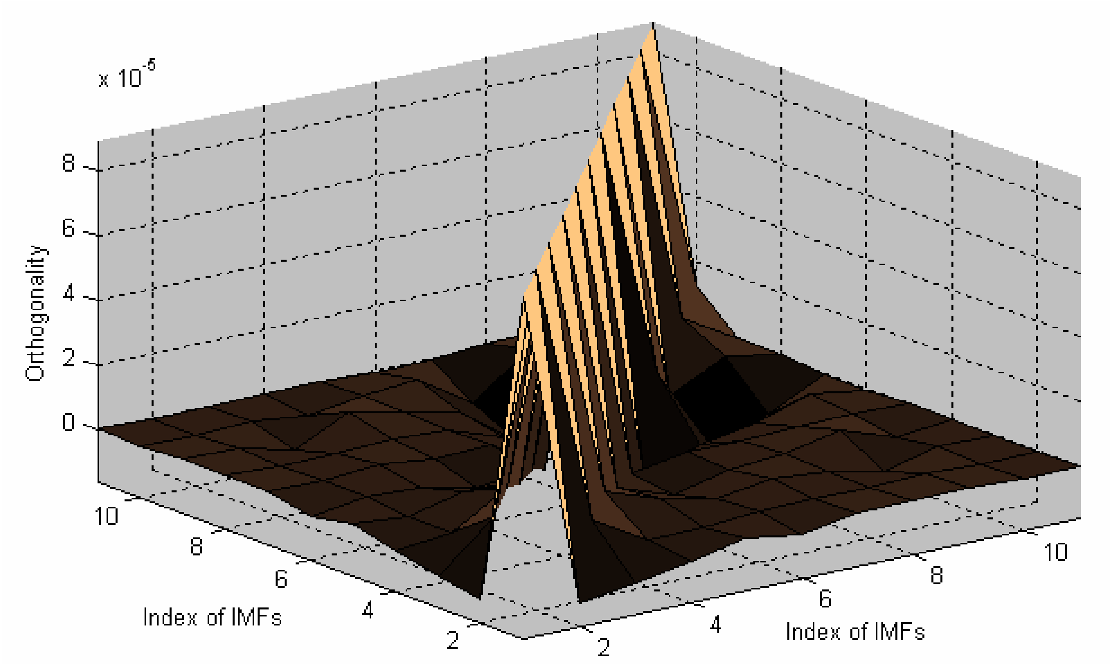

Figure 2.6a: Orthogonality between IMFs of maximum temperature

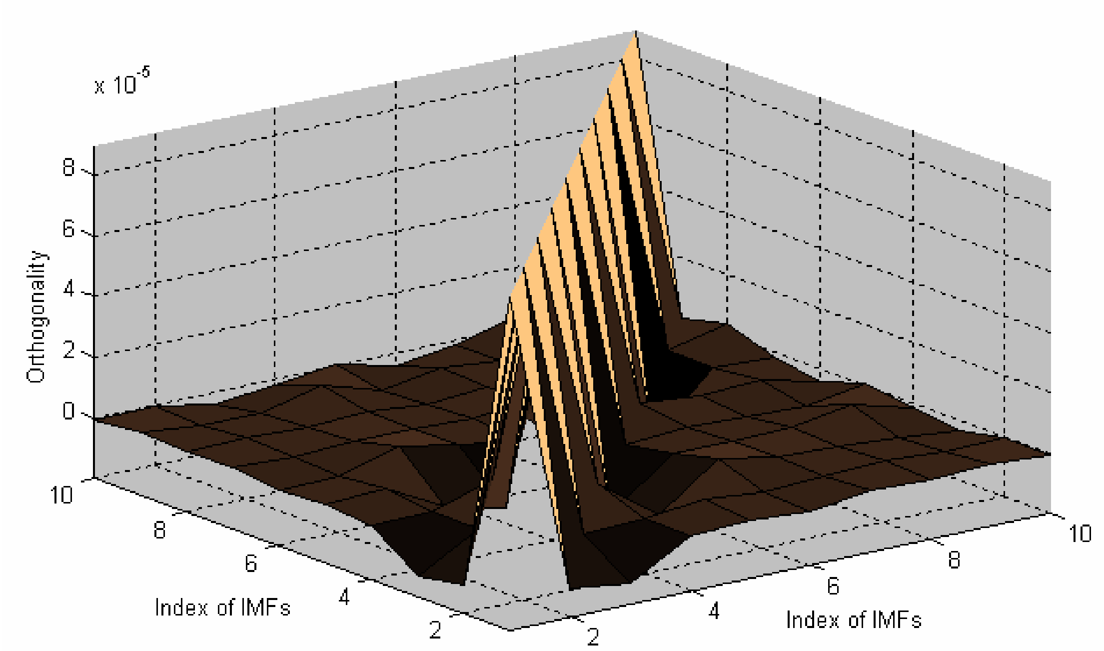

Figure 2.6b: Orthogonality between IMFs of minimum temperature

2.2.3. Some Parameters of IMF Components

Some parameters from IMF components can help us to analyze the 15 years daily temperature data given in Table 2.1 and Table 2.2. The mean period is calculated as the total number of extrema divided by the number of data samples. The percentage of energy content represents that how much energy each IMF contains. The variance indicates the amount if information of each IMF. The iteration offer the computational cost of each IMF. This paper will not discuss what these values mean in climate because it is highly relative to the meteorology.

Table 2.1: Some parameters of EMD for maximum temperature

IMF	Mean Period	% of Energy	% of Variance	No. of Iteration
IMF1	0.3161	6.5599	3.9329	83
IMF2	0.1604	7.3642	4.9056	58
IMF3	0.0890	7.2141	4.5844	51
IMF4	0.0482	8.4089	6.4040	20
IMF5	0.0273	5.6837	2.6696	36
IMF6	0.0132	9.4468	9.0876	17
IMF7	0.0050	20.9887	31.0227	15
IMF8	0.0030	20.7931	31.2409	7
IMF9	0.0016	6.9380	4.3836	16
IMF10	0.0009	4.7700	1.5088	7
IMF11	0.0004	1.8325	0.2599	12

Table 2.2: Some parameters of EMD for minimum temperature

IMF	Mean Period	% of Energy	% of Variance	No. of Iteration
IMF1	0.3144	5.6123	2.1344	39
IMF2	0.1599	4.8425	1.5340	252
IMF3	0.0931	5.6961	2.1701	20
IMF4	0.0483	5.3102	1.7632	21
IMF5	0.0244	4.8265	1.3867	19
IMF6	0.0109	6.8594	2.7849	24
IMF7	0.0039	29.4860	48.7564	12
IMF8	0.0021	25.1336	36.0538	11
IMF9	0.0012	5.8507	1.6981	7
IMF10	0.0005	6.3827	1.7184	4

3. Conclusion

Different from traditional analysis, such as Fourier transform, the Hilbert-Huang transform is developed for non-linear and non-stationary data, so it is should be data-dependent and adaptive. Although the Hilbert-Huang transform still lacks of complete theory, it is presently the best analysis method for non-linear and non-stationary systems. Empirical mode decomposition can be useful time series analysis tool, particularly for analyzing the climate record of the atmosphere beyond annual time scales. Global climate phenomenon is usually separated in time, so the time series analysis is more proper for climate than spatial methods. Furthermore, trends, instantaneous frequency and amplitude modulation, which come from IMF components, make EMD especially appropriate for climate. However, those predictions of climate by the Hilbert-Huang transform is not mature enough that there are still many natural disasters, such as floods, droughts, and heat waves, happening around the world. As a result, we still have many problems to conquer.

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