A New Formulation for Empirical Mode Decomposition Based on Constrained Optimization

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Abstract

The *empirical mode decomposition* (EMD) is an algorithmic construction which aims at decomposing a signal into several modes called *intrinsic mode functions*. In the current paper, we present a new approach for the EMD based on the direct construction of the mean envelope of the signal. The definition of the mean envelope is achieved through the resolution of a quadratic programming problem with equality and inequality constraints. Some numerical experiments conclude the paper and comparisons are carried out with the classical EMD.

Index Terms

Empirical Mode Decomposition, quadratic programming.

EDICS Category: TFSR

I. Introduction

Many methods exist to analyze signals in the time and frequency domain simultaneously, as, for instance, the short-time Fourier transform, the Wigner-Ville distribution and wavelet analysis [1][9]. The concept of the *empirical mode decomposition* (EMD) is to expand the signal into a set of functions defined by the signal itself, the *intrinsic mode functions* (IMF). The computation of the IMFs is achieved through a so-called ”sifting process” (SP) which is an iterative procedure whose study is particularly complicated.

The definition of the IMFs computed by the EMD is somewhat unclear. In the seminal work of Huang [7], it is suggested that the IMFs given by the SP are functions with zero local mean, i.e. their integral between two successive extrema is null. More recently, it was shown in [4], that this condition should not be fulfilled in many instances. A more accurate definition is then to define an IMF as a function having symmetric upper and lower envelopes. The problem with such a definition is that it makes the method depend on the interpolation method used to build the envelope. The condition of symmetric envelopes naturally leads to modes having positive maxima and negative minima. These **weak**-IMF functions have been studied in [10]. In that paper, it is shown that a **weak**-IMF can be viewed as a mono-component signal. Nevertheless, even if the EMD can be included in the general class of mono-component signal decompositions, the modes extraction is not based on frequency analysis, which makes the method very different from Fourier or wavelet based methods [3].

After we have recalled the classical definition for the EMD, we will introduce a new approach for the EMD. It will be based on the construction of the mean envelope through the resolution of a quadratic programming problem with equality and inequality constraints which we detail in section III. We will end the paper by comparing the new approach we propose to the traditional EMD algorithm.

II. EMD Basis

Let us first briefly recall the principle of the EMD for a one-dimensional signal $s[n]_{0\leq n\leq N-1}$.

1) Initialization : $r = s$, $k = 1$
2) Computation of the mean envelop e of r (i.e. the average of the envelope of the minima and of the maxima of r)
3) Extract the proto-mode function (PMF) $p_i = r - e$ and let $r = e$.
4) While p_i is not an IMF repeat
   - Compute the mean envelop e_i of p_i
   - $p_{i+1} = p_i - e_i$; $i = i + 1$

5) $d_k = p_i$, $r = r - d_k$

6) if r is not monotonic, go to step 2 and put $k = k + 1$; otherwise the decomposition is complete.

When the decomposition is complete, we can write s as follows:

$$s[n] = \sum_{k=1}^{K} d_k[n] + r[n].$$

The modes d_k are thus computed in a totally adaptive way. However, there is no mathematical proof that the obtained modes are with zero local mean as suggested by Huang [7]. Furthermore, as no proof of the convergence of the SP (corresponding to the point 4 in the algorithm) is available, the algorithm depends also on a stopping criterion. Different stopping criteria lead to different decompositions as we will see in numerical applications.

The first stopping criterion [7] was based on the comparison of the successive PMFs p_i obtained after i iterations of the SP:

$$SD = \sum_{n=0}^{N-1} \frac{(p_{i-1}[n] - p_i[n])^2}{p_{i-1}^2[n]}.$$

Since SD is unrelated to the definition of an IMF, the component obtained with this criterion cannot be an IMF. As an improvement of this stopping criterion, in [4], the authors define $a[n] = (e_{\max}[n] - e_{\min}[n])/2$ and $m[n] = (e_{\max}[n] + e_{\min}[n])/2$ and the evaluation function $\sigma[n] = |\frac{m[n]}{a[n]}|$. The sifting process carries on until $\sigma[n] < \theta_1$ for some special fraction $(1-\alpha)$ of the total duration, while $\sigma[n] < \theta_2$ for the remaining fraction. A typical value for $(\alpha, \theta_1, \theta_2)$ is $(0.05, 0.05, 0.5)$. One of the drawbacks of this method is that the thresholds do not adapt to the analyzed signal. This criterion assumes that the variations of an IMF are large compared to its mean envelope value most of the time, while these are roughly of the same order as the mean envelope value for the remaining times. This stopping criterion suggests that interesting modes have symmetric upper and lower envelope. By upper (resp. lower) envelope we mean the envelope of the maxima (resp. minima). This condition of symmetry is dependent on the choice of the interpolation method. In the following, we will exploit the idea that the upper and lower envelope must be symmetric and we drop the condition that a mode should have zero local mean. The alternative technique we will proposed will also enable us to eliminate the dependence of the EMD on an interpolation method.

III. A New Construction of the Mean Envelope based on Constrained Optimization

In this section, we propose a new construction of the mean envelope of a signal based on constrained optimization. The model chosen for the mean envelope is that of a piecewise cubic polynomial globally continuously differentiable. In the classical EMD algorithm, the construction of the upper (resp. lower) is achieved through a cubic spline interpolation of the maxima (resp. minima). Consequently, the mean envelope is also a cubic spline with knots defined by the set of extrema locations. Therefore, it is natural to consider that the mean envelope is a piecewise cubic polynomial. However, when one uses the cubic spline interpolation, overshoots and undershoots are common. We recall that an overshoot corresponds to an extremum of the envelope of the maxima that is larger than the global maximum of the signal (for an undershoot, the definition is similar but for the envelope of the minima). Overshoots sometimes give birth to numerical artifacts that propagate through the EMD. For that reason, we will not use the cubic spline interpolation in what follows.

A. *Definition of the Mean Envelope* e

Let us now detail our approach for the construction of the mean envelop e. Let us consider t_j, $j = 1, \cdots, J$ the increasing sequence of the locations of the extrema of the signal s. We seek an envelope which is a piecewise cubic polynomial on each interval $[t_j, t_{j+1}]$ and which is globally C^1. Therefore, $e(t) = \sum_{j=1}^{J-1} e_j(t)\chi_{[t_j,t_{j+1}[}(t)$, where $e_j(t) = a_j t^3 + b_j t^2 + c_j t + d_j$ and $\chi_{[t_j,t_{j+1}[}(t)$ is the characteristic function of $[t_j, t_{j+1}[$. The polynomials e_j are completely defined by the knowledge of $e_j(t_j)$, $e'_j(t_j)$, $e_j(t_{j+1})$ and $e'_j(t_{j+1})$. To get a C^1 envelope one must also impose $e_j(t_j) = e_{j-1}(t_j) = e(t_j)$ and $e'_j(t_j) = e'_{j-1}(t_j) = e'(t_j)$. The unknowns are thus the value of e and of e' at each extremum point t_j. Let us denote $\Lambda = [e(t_1), e'(t_1), e(t_2), e'(t_2), \cdots, e(t_{J-1}), e'(t_{J-1}), e(t_J), e'(t_J)]$, the vector of unknowns. The constraints we put on e are directly derived from the symmetry of the upper and lower envelope. We also pay attention to the fact that the number of extrema for the upper and lower envelope should be the same. To avoid the dependence on a particular choice of interpolation method, our approach is only based on the study of $s(t_j),\ j = 1, \cdots, J$.

B. *Inequality Constraints*

We here assume that the sequence $(s(t_{j-2}), s(t_j), s(t_{j+2}))$ is monotonic (or possibly constant) and that s has a minimum at t_j (the following reasoning still holds when s has a maximum at t_j). The symmetrical point P_j of $(t_j, s(t_j))$ with respect to $(t_j, e(t_j))$ is $(t_j, 2e(t_j) - s(t_j))$. In the classical EMD formulation, the mean envelope is defined as the average of the upper and lower envelope. P_j should therefore belong to the upper envelope. As we know that $(t_{j-1}, s(t_{j-1}))$ and $(t_{j+1}, s(t_{j+1}))$ belong to the upper envelope, to preserve the monotonicity of the data, we impose:

$$\begin{aligned} &\min(s(t_{j-1}), s(t_{j+1})) \leq 2e(t_j) - s(t_j) \leq \max(s(t_{j-1}), s(t_{j+1})) \\ \Leftrightarrow\ &\frac{\min(s(t_{j-1}), s(t_{j+1})) + s(t_j)}{2} \leq e(t_j) \leq \frac{\max(s(t_{j-1}), s(t_{j+1})) + s(t_j)}{2}. \end{aligned}$$

These conditions, together with the cases where $s(t_j)$ is a maximum, can be written as inequality constraints of the form $M_1\Lambda \leq S_1$ (upper bounds) and $N_1\Lambda \leq S_2$ (lower bounds). In that framework, $N_1 = -M_1$.

C. Equality Constraints

Now, we consider that $s(t_j)$ is an extremum for the sequence $(s(t_{j-2}), s(t_j), s(t_{j+2}))$. Depending on the cases, we use the shape of the upper (resp. lower) envelope to derive that of the lower (resp. upper).

Let $\tilde{t}_j$ be abscissa of the intersection (when it exists) of straight lines L_1 and L_2, defined by:

$$L_1: \; f_1(t) = \frac{s(t_j) - s(t_{j-2})}{t_j - t_{j-2}} t + \frac{s(t_{j-1})(t_j - t_{j-2}) - (s(t_j) - s(t_{j-2}))t_{j-1}}{t_j - t_{j-2}}$$

and

$$L_2: \; f_2(t) = \frac{s(t_{j+2}) - s(t_j)}{t_{j+2} - t_j} t + \frac{s(t_{j+1})(t_{j+2} - t_j) - (s(t_{j+2}) - s(t_j))t_{j+1}}{t_{j+2} - t_j}.$$

If $\tilde{t}_j > t_j$, we impose: $\frac{1}{2}\left(f_1(t_j) + s(t_j)\right) = e(t_j)$. Otherwise, we set $\frac{1}{2}\left(f_2(t_j) + s(t_j)\right) = e(t_j)$. These conditions can be written in the form $Q\Lambda = S_3$. Note that these equality and inequality constraints imply that $e(t_j) \leq$ (resp. $\geq$) $s(t_j)$ when there is a maximum (resp. minimum) at t_j. We also remark that, since the equality and inequality constraints are related to disjoint sets of extrema whose union is the whole set of extrema, the set of constraints is never empty as soon as the signal has at least one extremum.

D. Boundary Conditions

As the formulation we propose is based on the definition of the mean envelope defined as a piecewise cubic polynomial between extrema points, both the first and the last point have therefore to be extrema. We consequently symmetrize the initial signal with respect to its last and its first extremum so that the support of the envelope overlaps that of the original signal in such a way that the first and the last index of the new signal corresponds to an extremum.

E. Cost Function

As e_j can be viewed as the Hermite interpolant of e on $[t_j, t_{j+1}]$, it can therefore be written as:

$$e_j(t) = e(t_j)h_{j,j}(t) + e'(t_j)k_{j,j}(t) + e(t_{j+1})h_{j+1,j}(t) + e'(t_{j+1})k_{j+1,j}(t)$$

where $h_{j,l}(t) = \left(1 - 2(t - t_j)l'_{j,l}(t_j)\right) l^2_{j,l}(t)$ and $k_{j,l}(t) = (t - t_j)l^2_{j,l}(t)$, where $l_{j,l}$ is the Lagrange polynomial of degree 1 at t_j on the interval $[t_l, t_{l+1}]$, and l is in $\{j, j-1\}$.

As we seek an overall C^1 function, it is natural to consider the following cost function:

$$J(\Lambda) = \sum_{j=1}^{J-1} \int_{t_j}^{t_{j+1}} e_j'(t)^2 dt.$$

After some computations which we do not detail here due to the lack of space, we can then rewrite $J(\Lambda)$ as follows (with the convention that $t_0 = t_1$ and $t_{J+1} = t_J$ and putting $\delta t_j = t_{j+1} - t_j$):

$$\begin{aligned} J(\Lambda) = & \sum_{j=1}^{J-1} e^2(t_j)\frac{6}{5}\left(\frac{1}{\delta t_j} + \frac{1}{\delta t_{j-1}}\right) + e(t_j)e(t_{j+1})\left(-\frac{12}{5\delta t_j}\right) + e(t_j)e'(t_{j+1})\left(\frac{1}{5}\right) \\ & + e'(t_j)^2\left(\frac{2}{15}(\delta t_j + \delta t_{j-1})\right) + e'(t_j)e(t_{j+1})\left(-\frac{1}{5}\right) + e'(t_j)e'(t_{j+1})\left(\frac{\delta t_j}{15}\right) \\ & + e^2(t_J)\left(\frac{6}{5\delta t_{J-1}}\right) + e(t_J)e'(t_J)\left(-\frac{1}{5}\right) + e'(t_J)^2\left(\frac{2}{15}\delta t_{J-1}\right) \end{aligned}$$

From this, we can deduce that $J(\Lambda) = \Lambda^T B \Lambda$, where B is a symmetric matrix, which is:

$$B = \begin{pmatrix} B_1 & C_1^T & 0 & \cdots & 0 \\ C_1 & B_2 & C_2^T & \ddots & \vdots \\ 0 & C_2 & B_3 & C_3^T & 0 \\ \vdots & \ddots & \ddots & \ddots & C_{J-1}^T \\ 0 & \cdots & 0 & C_{J-1} & B_J \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} \frac{6}{5\delta t_1} & \frac{1}{10} \\ \frac{1}{10} & \frac{2\delta t_1}{15} \end{pmatrix}, \quad B_J = \begin{pmatrix} \frac{6}{5\delta t_{J-1}} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2\delta t_{J-1}}{15} \end{pmatrix}$$

and for $j = 2, \cdots, J-2$,

$$B_j = \begin{pmatrix} \frac{6}{5}\left(\frac{1}{\delta t_{j-1}} + \frac{1}{\delta t_j}\right) & 0 \\ 0 & \frac{2}{15}(\delta t_j + \delta t_{j-1}) \end{pmatrix}.$$

Also, for $j = 1, \cdots, J-1$, $C_j = \begin{pmatrix} -\frac{6}{5\delta t_j} & -\frac{1}{10} \\ \frac{1}{10} & \frac{\delta t_j}{30} \end{pmatrix}$

The problem of the construction of the mean envelope we solve is thus:

$$\left\{ \begin{array}{c} \min_{\Lambda} \Lambda^T B \Lambda \\ M_1\Lambda \leq S_1,\ N_1\Lambda \leq S_2,\ Q\Lambda = S_3 \end{array} \right. ,$$

which is a classical problem of quadratic programming. As B is positive semidefinite matrix, J is a convex function. In this case, since $J(\Lambda)$ is bounded below, the quadratic program has a global minimizer if there exists at least one vector satisfying the constraints, which is always the case provided the signal has at least one extremum. Now, if Λ_0 is a minimum for J, it satisfies the Euler inequality: $\Lambda_0^T B(\Lambda - \Lambda_0) \geq 0\ \forall \Lambda \in Y$, where Y is the set of constraints. Now assume that Λ_1 is another minimum then it also satisfies the Euler inequality. We then derive from this $(\Lambda_0-\Lambda_1)^T B(\Lambda_0-\Lambda_1) \leq 0$ which leads to $(\Lambda_0-\Lambda_1)^T B(\Lambda_0-\Lambda_1) = 0$, since B is positive semidefinite matrix. The definition of J implies that the piecewise cubic polynomial associated to $\Lambda_0 - \Lambda_1$ is a constant function. Consequently, if e, a piecewise cubic polynomial associated to a minimum for J, has a fixed value imposed by the constraints at one t_j then the minimum will be unique. First, let us assume that $s(t_j) = s(t_{j+2})$ for some j, then the inequality constraints fix the value of e at t_j. Second, if $s(t_j)$ is an extremum for $(s(t_{j-2}), s(t_j), s(t_{j+2}))$, the equality constraints fix the value of e at t_j. Finally, assume that both the sequences of maxima and of minima are strictly monotonic. In such a case, one can build examples where there are several minima for J satisfying the constraints corresponding to functions that are equal up to a constant. However, the boundary conditions, fixing the value of $e(t_j)$ for some j, ensures the uniqueness of the minimum. To conclude, in any case, the minimum is unique.

It is proved [8] that the quadratic optimization problem can be solved in polynomial time. Many different methods are available to solve this kind of problem among which the class of interior methods [6] is the most popular. To compute Λ^* we will use the *quapro* routine of scilab [11].

IV. Numerical Applications

We will now show that we obtain similar results with our method than with the classical EMD defined by the algorithm of section II (we use the program detailed in [4]). We will then conclude that we can profitably use the mathematically well settled method we propose instead of the classical EMD which lacks a clear mathematical framework. We will now present some numerical results on a nonstationary signal and then on a stationary one.

A. The Nonstationary Case

Let us consider the signal $s(t) = \sin(t^2)\sin(2k\pi t)$ where k is an integer larger than one. In the classical EMD approach, the first mode is nearly equal to the signal s, when k is chosen sufficiently large. However, with that approach it is difficult to give a characterization of the remaining signal (i.e. s minus the first mode). Furthermore, depending on the number of iterations in the SP, the first mode can be significantly different. To the contrary, with our approach and when k varies, the first mode will always be equal to the signal itself. These facts are illustrated in Figure 1: for two values $k = 2$ (Figure 1.A) and $k = 5$ (Figure 1.B) we display the signal and the first mode either obtained with the classical EMD approach (M_C), or with the approach we propose (M_O). With our approach we obtain the first

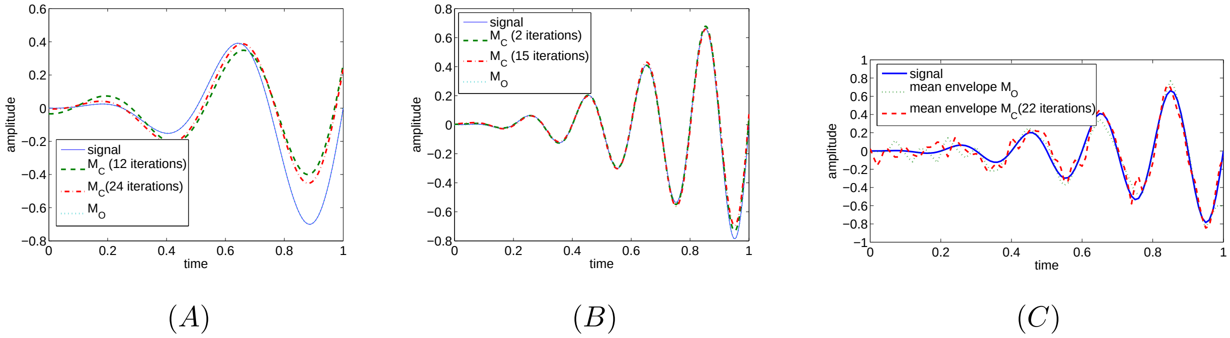


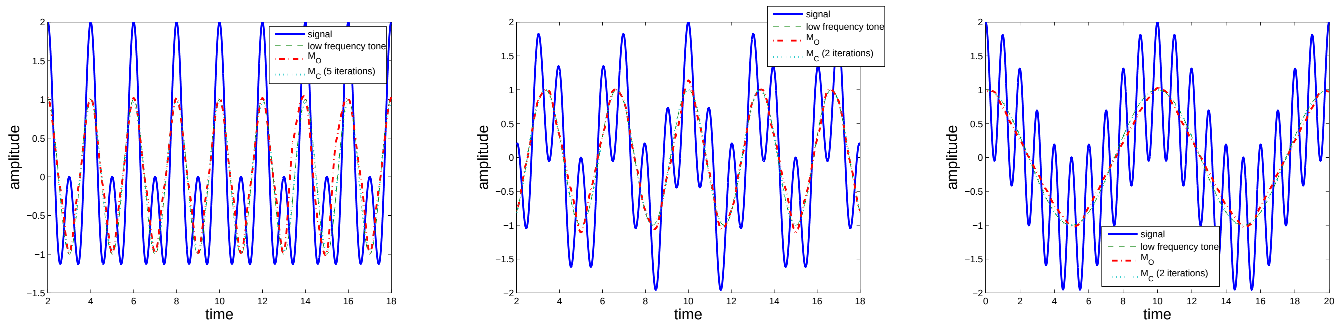
Fig. 1. (A): computation of the first mode when $k = 2$ either with the classical EMD algorithm (M_C) or with the constrained optimization approach we propose (M_O),(B): same except for $k = 5$,(C): mean envelope for $k = 5$ with the two methods and the noise-free original signal

mode subtracting the mean envelope to the signal. Note that in such a case, the mode is indistinguishable from the signal and that only the inequality constraints are taken into account since the envelope of the minima is decreasing and the envelope of the maxima is increasing.

If we now add some noise to the signal, we expect that the mean envelope of the noisy signal will look the same as the noise-free original signal. This is indeed the case with the method we propose and with the classical EMD algorithm (see Figure 1 (C), SNR = -9,28 dB). When the amplitude of the noise is of the same order as that of the signal, the mean envelopes become more oscillatory (see the left part of Figure 1 (C)). The study of noisy signals with EMD would require further developments but this is beyond the scope of the present article.

B. The Stationary Case

Let us now consider the family of signals $s(t) = \cos(2\pi t) + a\cos(2\pi f t)$, with $f \in (0, 1)$ and $a > 0$. In [5], it is shown that with such a signal when $0 < f \leq 0.5$ and $0 < a < 1$, the separation between the two tones is perfect with the EMD. Furthermore, if, given f, the separation is successful for a given a, it will also be successful for a smaller a. We therefore compare the envelopes given by the classical EMD and by our method for $a = 1$ and varying f, f being inferior to 0.5. In Figure 2, we display the signal, the low frequency tone (i.e. $\cos(2\pi f t)$) and the mean envelope computed either as in [4] (M_C in Figure 2 ; the envelope is obtained by subtraction of the first mode to the signal) or by the new technique we propose (M_O in Figure 2). Figures 2 (A),(B) and (C) correspond to $f = 0.1, 0.3$ and 0.5 respectively. For the method M_C, the number of iterations in the SP is 2 when $f = 0.1$ or 0.3 and 5 when $f = 0.5$. Note



(A) (B) (C)

Fig. 2. (A): the signal $s(t) = \cos(2\pi t) + \cos(2\pi f t)$, for $f = 0.5$, the low frequency tone (i.e.$\cos(2\pi f t)$) and the mean envelope obtained either with our method (M_O) or with the classical EMD approach (M_C);(B): same except for $f = 0.3$(C): same except for $f = 0.1$

that to choose a different stopping criterion would change the mean envelope. On the contrary, with the method we propose the low frequency tone is well retrieved and the proposed algorithm does not need any stopping criterion.

V. Conclusion

In this paper, we have proposed an alternative method to the classical algorithmic approach for the EMD. The results we obtained are very similar to these obtained with the traditional algorithm. In that context, the mean envelope (and consequently the modes) are given by the resolution of a quadratic programming problem which enables us to alleviate the dependence on a stopping criterion of the traditional approach. We are currently pursuing our research towards the extension of the method to the bidimensional case as it was done with the classical approach in [2].

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